Sparse Logistic Regression Learns All Discrete Pairwise Graphical Models

Shanshan Wu, Sujay Sanghavi, Alex Dimakis

University of Texas at Austin
Graphical models are used to describe complex dependency structures.

Social network analysis
Natural language processing
Biology


Discrete pairwise graphical model

- **Binary case** (aka Ising model):
  - An undirected graph on $n$ nodes
  - $\mathbb{P}[Z = z] \propto \exp(\sum_{1\leq i<j\leq n} A_{ij}z_i z_j + \sum_{i \in [n]} \theta_i z_i)$

- **Non-binary case** (alphabet size $k$): $Z \in \{1,2,\ldots,k\}^n$
The structure learning problem

**Given:** i.i.d. samples from an unknown graphical model

\[
\begin{array}{cccccccc}
Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & \ldots & Z_n \\
\hline
\text{Sample 1} & [-1 & 1 & -1 & -1 & 1 & \ldots & 1] \\
\text{Sample 2} & [1 & -1 & -1 & 1 & -1 & -1 & \ldots & 1] \\
\ldots & \ldots
\end{array}
\]

**Goal:** Recover the graph, i.e., identify the edges

\[
\begin{array}{c}
Z_i \quad \text{\textbullet} \quad Z_j
\end{array}
\]

A simple approach...

Maximize the conditional log-likelihood

Binary case

$\ell_1$-regularized logistic regression
[Ravikumar et al.'10]

Non-binary case

$\ell_{2,1}$-regularized logistic regression
[Jalali et al.'11]
Limitation of [Ravikumar et al.’10, Jalali et al.’11]

Assuming that the graphical models satisfy an incoherence condition,

sparse logistic regression provably recover the graph structure.
Our contribution

Assuming that the graphical models satisfy an incoherence condition,
For all graphical models,
sparse logistic regression provably recover the graph structure.
Our contribution

- Let $n = \#\text{ variables}$, alphabet size $k$, width $\lambda$, minimum edge weight $\eta$

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Sample complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greedy algorithm [Hamilton et al.'17]</td>
<td>$O(\exp\left(\frac{k^{O(d)} \exp(d^2 \lambda)}{\eta^{O(1)}}\right) \ln(nk))$</td>
</tr>
<tr>
<td>Sparsitron [Klivans and Meka’17]</td>
<td>$O\left(\frac{\lambda^2 k^5 \exp(14 \lambda)}{\eta^4} \ln \left(\frac{nk}{\eta}\right)\right)$</td>
</tr>
<tr>
<td>$\ell_{2,1}$-constrained logistic regression [Our work]</td>
<td>$O\left(\frac{\lambda^2 k^4 \exp(14 \lambda)}{\eta^4} \ln (nk)\right)$</td>
</tr>
</tbody>
</table>

Improves from $k^5$ to $k^4$!
Sparse logistic regression requires fewer samples for graph recovery.
Poster #183

Today 10:45 AM -- 12:45 PM

@East Exhibition Hall B + C